
Introduction to Problem-Solving Strategies

Before we can discuss what problem solving is, we must first come to grips with what is meant by a problem. In essence, a problem is a situation that confronts a person, that requires resolution, and for which the path to the solution is not immediately known. In everyday life, a problem can manifest itself as anything from a simple personal problem, such as the best strategy for crossing the street (usually done without much “thinking”), to a more complex problem, such as how to assemble a new bicycle. Of course, crossing the street may not be a simple problem in some situations. For example, Americans become radically aware of what is usually a subconscious behavior pattern while visiting a country such as England, where their usual strategy for safely crossing the street just will not work. The reverse is also true; the British experience similar feelings when visiting the European continent, where traffic is oriented differently than that in Britain. These everyday situations are usually resolved “subconsciously,” without our taking formal note of the procedures by which we found the solution. A consciousness of everyday problem-solving methods and strategies usually becomes more evident when we travel outside of our usual cultural surroundings. There the usual way of life and habitual behaviors may not fit or may not work. We may have to consciously adapt other methods to achieve our goals.

Much of what we do is based on our prior experiences. As a result, the level of sophistication with which we attack these problems will vary with the individual. Whether the problems we face in everyday life involve selecting a daily wardrobe, relating to friends or acquaintances, or dealing with professional issues or personal finances, we pretty much function automatically, without considering the method or strategy that best suits the situation. We go about addressing life's challenges with an algorithmic-like approach and can easily become a bit frustrated if that approach suddenly doesn't fit. In these situations, we are required to find a solution to the problem. That is, we must search our previous experiences to find a way we solved an analogous problem in the past. We could also reach into our bag of problem-solving tools and see what works.

When students encounter problems in their everyday school lives, their approach is not much different. They tend to tackle problems based on their previous experiences. These experiences can range from recognizing a "problem" as very similar to one previously solved to taking on a homework exercise similar to exercises presented in class that day. The student is not doing any problem solving—rather, he or she is merely mimicking (or practicing) the earlier encountered situations. This is the behavior seen in a vast majority of classrooms. In a certain sense, repetition of a "skill" is useful in attaining the skill. This can also hold true for attaining problem-solving skills. Hence, we provide ample examples to practice the strategy applications in a variety of contexts.

This sort of approach to dealing with what are often seen as artificial situations, created especially for the mathematics class, does not directly address the idea of problem solving as a process to be studied for its own sake, and not merely as a facilitator. People do not solve "age problems," "motion problems," "mixture problems," and so on in their real lives. Historically, we always have considered the study of mathematics topically. Without a conscious effort by educators, this will clearly continue to be the case. We might rearrange the topics in the syllabus in various orders, but it will still be the topics themselves that link the courses together rather than the mathematical procedures involved, and this is not the way that most people think! Reasoning involves a broad spectrum of thinking. We hope to encourage this thinking here.

We believe that there can be great benefits to students in a mathematics class (as well as a spin-off effect in their everyday lives) by considering problem solving as an end in itself and not merely as a means to an end. Problem solving can be the vehicle used to introduce our students to the beauty that is inherent in mathematics, but it can also be the unifying thread that ties their mathematics experiences together into a meaningful whole. One immediate goal is to have our students become familiar with numerous problem-solving strategies and to practice using them. We expect this procedure will begin to show itself in the way students approach problems and ultimately solve them. Enough practice of this kind should, for the most part, make a longer-range goal attainable, namely, that students

naturally come to use these same problem-solving strategies not only to solve mathematical problems but also to resolve problems in everyday life. This transfer of learning (back and forth) can be best realized by introducing problem-solving strategies in both mathematical and real-life situations concomitantly. This is a rather large order and an ambitious goal as well. Changing an instructional program by relinquishing some of its time-honored emphasis on isolated topics and concepts, and devoting the time to a procedural approach, requires a great deal of teacher "buy-in" to succeed. This must begin by convincing the teachers that the end results will prepare a more able student for this era, where the ability to think becomes more and more important as we continue to develop and make use of sophisticated technology.

When we study the history of mathematics, we find breakthroughs that, although simple to understand, often elicit the reaction, "Oh, I would never have thought about that approach." Analogously, when clever solutions to certain problems are found and presented as "tricks," they have the same effect as the great breakthroughs in the history of mathematics. We must avoid this sort of rendition and make clever solutions part of an attainable problem-solving strategy knowledge base that is constantly reinforced throughout the regular instructional program.

You should be aware that, in the past few decades, there has been much talk about problem solving. While many new thrusts in mathematics last a few years, then disappear leaving some traces behind to enrich our curriculum, the problem-solving movement has endured for more than a quarter of a century and shows no sign of abatement. If anything, it shows signs of growing stronger. The National Council of Teachers of Mathematics (NCTM), in its *Agenda for Action* (1980), firmly stated that "problem solving must be the focus of the (mathematics) curriculum." In their widely accepted *Curriculum and Evaluation Standards for School Mathematics* (1989), the NCTM offered a series of process Standards, in addition to the more traditional content Standards. Two of these four Standards (referred to as the "Process Standards"), Problem Solving and Reasoning, were for students in all grades, K through 12. In their *Principles and Standards for School Mathematics* (2000), the NCTM continued this emphasis on problem solving throughout the grades as a major thrust of mathematics teaching. All these documents have played a major role in generating the general acceptance of problem solving as a major curricular thrust. Everyone seems to agree that problem solving and reasoning are, and must be, an integral part of any good instructional program. In an effort to emphasize this study of problem solving and reasoning in mathematics curricula, most states are now including problem-solving skills on their statewide tests. Teachers sometimes ask, "If I spend time teaching problem solving, when will I find the time to teach the arithmetic skills the children need for the state test?" In fact, research has shown that students who are taught via a problem-solving mode of instruction usually do as well, or better, on state tests than many other students who have spent all

their time learning only the skills. After all, when solving a problem, one must dip into his or her arsenal of arithmetic skills to find the correct answer to the problem. Then why has the acceptance of problem solving as an integral part of the mathematics curriculum not come to pass? In our view, the major impediment to a successful problem-solving component in our regular school curriculum is a weakness in the training teachers receive in problem solving, as well as the lack of attention paid to the ways in which these skills can be smoothly incorporated into their regular teaching program. Teachers ought not to be forced to rely solely on their own resourcefulness as they attempt to move ahead without special training. They need to focus their attention on what problem solving is, how they can use problem solving to teach the skills of mathematics, and how problem solving should be presented to their students. They must understand that problem solving can be thought of in three different ways:

1. Problem solving is a subject for study in and of itself.
2. Problem solving is an approach to a particular problem.
3. Problem solving is a way of teaching.

Above all, teachers must focus their attention on their own ability to become competent problem solvers. It is imperative that they know and understand problem solving if they intend to be successful when they teach it. They must learn which problem-solving strategies are available to them, what these entail, and when and how to use them. They must then learn to apply these strategies, not only to mathematical situations but also to everyday life experiences whenever possible. Often, simple problems can be used in clever ways to demonstrate these strategies. Naturally, more challenging problems will show the power of the problem-solving strategies. By learning the strategies, beginning with simple applications and then progressively moving to more challenging and complex problems, the students will have opportunities to grow in the everyday use of their problem-solving skills. Patience must be used with students as they embark on, what is for most of them, this new adventure in mathematics. We believe that only after teachers have had the proper immersion in this alternative approach to mathematics in general and to problem solving in particular, and after they have developed sensitivity toward the learning needs and peculiarities of students, then, and only then, can we expect to see some genuine positive change in students' mathematics performance.

We will set out with an overview of those problem-solving strategies that are particularly useful as tools in solving mathematical problems. From the outset, you should be keenly aware that it is rare that a problem can be solved using all 10 strategies we present here. Similarly, it is equally rare that only a single strategy can be used to solve a given problem. Rather, a combination of strategies is the most likely occurrence when solving a problem. Thus, it is best to become familiar with all the strategies

and to develop facility in using them when appropriate. The strategies selected here are not the only ones available, but they represent those most applicable to mathematics instruction in the schools. The user will, for the most part, determine appropriateness of a strategy in a particular problem. This is analogous to carpenters, who, when called on to fix a problem with toolbox in hand, must decide which tool to use. The more tools they have available and the better they know how to use them, the better we would expect the results to be. However, just as not every task carpenters have to do will be possible using the tools in their toolbox, so, too, not every mathematics problem will be solvable using the strategies presented here. In both cases, experience and judgment play an important role.

We believe that every teacher, if he or she is to help students learn and use the strategies of problem solving, must have a collection from which to draw examples. Throughout the book, we make a conscious effort to label the strategies and to use these labels as much as possible so that they can be called on quickly, as they are needed. This is analogous to the carpenter deciding which tool to use in constructing something; usually, the tool is referred to by name (i.e., a label). For you to better understand the strategies presented in this book, we begin each section with a description of a particular strategy, apply it to an everyday problem situation, and then present examples of how it can be applied in mathematics. We follow this with a series of mathematics problems from topics covered in the schools, which can be used with your students to practice the strategy. In each case, the illustrations are not necessarily meant to be typical but are presented merely to best illustrate the use of the particular strategy under discussion. The following strategies will be considered in this book:

1. Working backwards
2. Finding a pattern
3. Adopting a different point of view
4. Solving a simpler, analogous problem (specification without loss of generality)
5. Considering extreme cases
6. Making a drawing (visual representation)
7. Intelligent guessing and testing (including approximation)
8. Accounting for all possibilities (exhaustive listing)
9. Organizing data
10. Logical reasoning

As we have already mentioned, there is hardly ever one unique way to solve a problem. Some problems lend themselves to a wide variety of solution methods. As a rule, students should be encouraged to consider

alternative solutions to a problem. This usually means considering classmates' solutions and comparing them with the "standard" solution (i.e., the one given in the textbook or supplied by the teacher). Indeed, it has been said that it is far better to solve one problem in four ways than to solve four problems, each in one way. In addition, we must again state that many problems may require more than one strategy for solution. Furthermore, the data given in the problem statement, rather than merely the nature of the problem, can also determine the best strategy to be used in solving the problem. All aspects of a particular problem must be carefully inspected before embarking on a particular strategy.

Let's consider a problem that most people can resolve by an intuitive (or random) trial-and-error method, but that might take a considerable amount of time. To give you a feel for the use of these problem-solving strategies, we will approach the problem by employing several of the strategies listed.

Problem 1.1

Place the numbers from 1 through 9 into the grid below so that the sum of each row, column, and diagonal is the same. (This is often referred to as a magic square.)

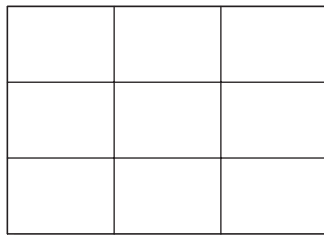


Figure 1.1

Solution

A first step to a solution would be to use *logical reasoning*. The sum of the numbers in all nine cells would be $1 + 2 + 3 + \cdots + 7 + 8 + 9 = 45$. If each row has to have the same sum, then each row must have a sum of $\frac{45}{3}$ or 15.

The next step might be to determine which number should be placed in the center cell. Using *intelligent guessing and testing* along with some additional *logical reasoning*, we can begin by trying some *extreme cases*. Can 9 occupy the center cell? If it did, then 8 would be in some row, column, or diagonal along with the 9, making a sum greater than 15. Therefore, 9 cannot be in the center cell. Similarly, 6, 7, or 8 cannot occupy the center cell, because then they would be in the same row, column, or diagonal with 9 and would not permit a three number sum of 15. Consider now the other extreme. Could 1 occupy the center cell? If it did, then it would be in

some row, column, or diagonal with 2, thus requiring a 12 to obtain a sum of 15. Similarly, 2, 3, or 4 cannot occupy the center cell. Having *accounted for all the possibilities*, this leaves only the 5 to occupy the center cell.

| | | |
|--|---|--|
| | | |
| | 5 | |
| | | |

Figure 1.2

Now, using *intelligent guessing and testing*, we can try to put the 1 in a corner cell. Because of symmetry, it does not matter which corner cell we use for this guess. In any case, this forces us to place the 9 in the opposite corner, if we are to obtain a diagonal sum of 15.

| | | |
|---|---|---|
| 1 | | |
| | 5 | |
| | | 9 |

Figure 1.3

With a 9 in one corner, the remaining two numbers in the row with the 9 must total 6; that is, 2 and 4. One of those numbers (the 2 or the 4) would then also be in a row or a column with the 1, making a sum of 15 impossible in that row or column. Thus, 1 cannot occupy a corner. Placing it in a middle cell of one outside row or column forces the 9 into the opposite cell so as to get a sum of 15.

| | | |
|--|---|--|
| | 1 | |
| | 5 | |
| | 9 | |

Figure 1.4

The 7 cannot be in the same row or column with the 1, because a second 7 would then be required to obtain a sum of 15.

| | | |
|---|---|---|
| 7 | 1 | ? |
| | 5 | |
| | 9 | |

Figure 1.5

In this way, we can see that 8 and 6 must be in the same row or column (and at the corner positions, of course) with the 1.

| | | |
|---|---|---|
| 8 | 1 | 6 |
| | 5 | |
| | 9 | |

Figure 1.6

This then determines the remaining two corner cells (4 and 2) to allow the diagonals to have a sum of 15:

| | | |
|---|---|---|
| 8 | 1 | 6 |
| | 5 | |
| 4 | 9 | 2 |

Figure 1.7

To complete the magic square, we simply place the remaining two numbers, 3 and 7, into the two remaining cells to get sums of 15 in the first and third columns.

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

Figure 1.8

In this solution to the problem, observe how the various strategies were used for each step of the solution.

We stated earlier that problems can (and should) be solved in more than one way. Let's examine an alternative approach to solving this same problem. Picking up the solution from the point at which we established that the sum of every row, column, or diagonal is 15, list all the possibilities of three numbers from this set of nine that have a sum of 15 (*accounting for all the possibilities*). By *organizing the data* in this way, the answer comes rather quickly:

| | | |
|--|---|--|
| | | |
| | 5 | |
| | | |

Figure 1.9

We now *adopt a different point of view* and consider the position of a cell and the number of times it is counted into a sum of 15 (*logical reasoning*). The center square must be counted four times: twice in the diagonals and once each for a row and a column. The only number that appears four times in the triples we have listed below is 5. Therefore, it must belong in the center cell.

| | |
|---------|---------|
| 1, 5, 9 | 2, 6, 7 |
| 1, 6, 8 | 3, 4, 8 |
| 2, 4, 9 | 3, 5, 7 |
| 2, 5, 8 | 4, 5, 6 |

The corner cells are each used three times. Therefore, we place the numbers used three times (the even numbers, 2, 4, 6, and 8) in the corners.

| | | |
|---|---|---|
| 8 | | 6 |
| | 5 | |
| 4 | | 2 |

Figure 1.10

The remaining numbers (the odd numbers) are each used twice in the above sums and, therefore, are to be placed in the peripheral center cells (where they are only used by two sums) to complete our magic square:

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

Figure 1.11

This *logical reasoning* was made considerably simpler by using a *visual representation* of the problem. It is important to have students realize that we solved the same problem in two very different ways. They should try to develop other alternatives to these, and they might also consider using consecutive numbers other than 1 to 9. An ambitious student might also consider the construction of a 4×4 or a 5×5 magic square.

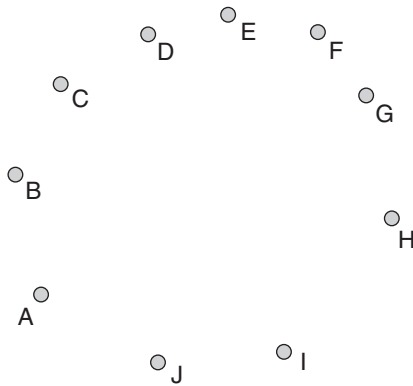
As we stated before, it is extremely rare to find a single problem that can be efficiently solved using each of the 10 problem-solving strategies we listed. There are times, however, when more than one strategy can be used, either alone or in combination, with varying degrees of efficiency. Of course, the level of efficiency of each method may vary with the reader. Let's take a look at one such problem. It's a problem that is well known, and you may have seen it before. We intend, however, to approach its solution with a variety of different strategies.

Problem 1.2

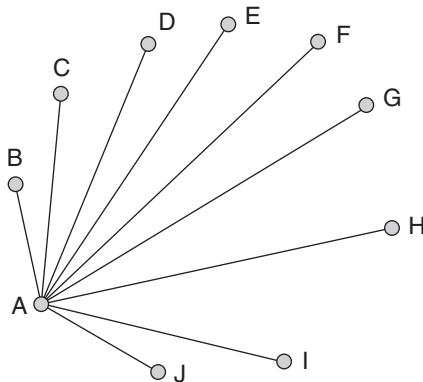
In a room with 10 people, everyone shakes hands with everybody else exactly once. How many handshakes are there?

Solution A

Let's use our *visual representation* strategy, by drawing a diagram. The 10 points (no 3 points of which are collinear) represent the 10 people. Begin with the person represented by point A.

**Figure 1.12**

We join A to each of the other 9 points, indicating the first 9 handshakes that take place.

**Figure 1.13**

Now, from B there are 8 additional handshakes (since A has already shaken hands with B and \overline{AB} is already drawn). Similarly, from C there will be 7 lines drawn to the other points (\overline{AC} and \overline{BC} are already drawn), from D there will be 6 additional lines or handshakes, and so on. When we reach point I , there is only one remaining handshake to be made, namely, I with J , since I has already shaken hands with $A, B, C, D, E, F, G,$ and H . Thus, the sum of the handshakes equals $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$. In general, this is the same as using the formula for the sum of the first n natural numbers, $\frac{n(n+1)}{2}$, where $n \geq 2$. (Notice that the final drawing will be a decagon with all its diagonals drawn.)

Solution B

We can approach the problem by *accounting for all the possibilities*. Consider the grid shown in Figure 1.14, which indicates persons A, B, C, \dots, H, I, J shaking hands with one another. The diagonal with the Xs indicates that people cannot shake hands with themselves.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| A | X | | | | | | | | | |
| B | | X | | | | | | | | |
| C | | | X | | | | | | | |
| D | | | | X | | | | | | |
| E | | | | | X | | | | | |
| F | | | | | | X | | | | |
| G | | | | | | | X | | | |
| H | | | | | | | | X | | |
| I | | | | | | | | | X | |
| J | | | | | | | | | | X |

Figure 1.14

The remaining cells indicate doubly all the other handshakes (i.e., A shakes hands with B , and B shakes hands with A). Thus, we take the total number of cells (10^2) minus those on the diagonal (10) and divide the result by 2. In this case, we have $\frac{100-10}{2} = 45$.

In a general case for the $n \times n$ grid, the number would be $\frac{n^2-n}{2}$, which is equivalent to the formula $\frac{n(n-1)}{2}$.

Solution C

Let's now examine the problem by *adopting a different point of view*. Consider the room with 10 people, each of whom will shake 9 other people's hands. This seems to indicate that there are 10×9 or 90 handshakes, but we must divide by 2 to eliminate the duplication (since when A shakes hands with B , we may also consider that as B shaking hands with A); hence, $\frac{90}{2} = 45$.

Solution D

Let's try to solve the problem by *looking for a pattern*. In the table shown in Figure 1.15, we list the number of handshakes occurring in a room as the number of people increases.

| <i>Number of People in Room</i> | <i>Number of Handshakes for Additional Person</i> | <i>Total Number of Handshakes in Room</i> |
|-------------------------------------|---|---|
| 1 | 0 | 0 |
| 2 | 1 | 1 |
| 3 | 2 | 3 |
| 4 | 3 | 6 |
| 5 | 4 | 10 |
| 6 | 5 | 15 |
| 7 | 6 | 21 |
| 8 | 7 | 28 |
| 9 | 8 | 36 |
| 10 | 9 | 45 |

Figure 1.15

The third column, which is the total number of handshakes, gives a sequence of numbers known as the *triangular numbers*, whose successive differences increase by 1 each time. It is therefore possible to simply continue the table until we reach the corresponding sum for the 10 people. Alternatively, we note that the pattern at each entry is one half the product of the number of people on that line and the number of people on the previous line.

Solution E

We can approach the problem by a careful use of the *organizing data* strategy. The chart in Figure 1.16 shows each of the people in the room and the number of hands they have to shake each time, given that they have already shaken the hands of their predecessors and don't shake their own hands. Thus, person number 10 shakes 9 hands, person number 9 shakes 8 hands, and so on, until we reach person number 2, who only has one person's hand left to shake, and person number 1 has no hands to shake because everyone already shook his hand. Again the sum is 45.

| <i>Organizing Data</i> | | | | | | | | | | |
|------------------------|----|---|---|---|---|---|---|---|---|---|
| No. of people | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| No. of handshakes | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

Figure 1.16

Solution F

We may also combine *solving a simpler problem with visual representation (drawing a picture), organizing the data, and looking for a pattern.* Begin by considering a figure with 1 person, represented by a single point. Obviously, there will be 0 handshakes. Now, expand the number of people to 2, represented by 2 points. There will be 1 handshake. Again, let’s expand the number of people to 3. Now, there will be 3 handshakes needed. Continue with 4 people, 5 people, and so on.

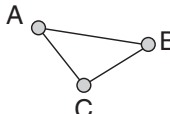
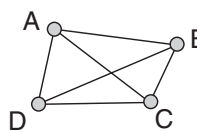
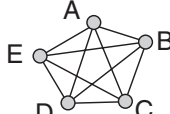
| <i>Number of People</i> | <i>Number of Handshakes</i> | <i>Visual Representation</i> |
|-------------------------|-----------------------------|--|
| 1 | 0 | ○A |
| 2 | 1 | A — B |
| 3 | 3 |  |
| 4 | 6 |  |
| 5 | 10 |  |

Figure 1.17

The problem has now become a geometry problem, in which the answer is the number of sides and diagonals of an “*n*-gon.” Thus, for 10 people we

have a decagon, and the number of sides, $n = 10$. For the number of diagonals, we may use the formula

$$d = \frac{n(n-3)}{2}, \text{ where } n > 3.$$

Hence,

$$d = \frac{(10)(7)}{2} = 35.$$

Thus, the number of handshakes equals $10 + 35 = 45$.

Solution G

Of course, some students might simply recognize that this problem could be resolved easily by applying the combinations formula of 10 things taken 2 at a time:

$${}_{10}C_2 = \frac{10 \times 9}{1 \times 2} = 45.$$

Although this solution is quite efficient, brief, and correct, it uses hardly any mathematical thought (other than application of a formula), and it avoids the problem-solving approach entirely. Although it is a solution that should be discussed, we must call the other solutions to the students' attention.

Notice that we continually differentiate between the terms *answer* and *solution*. The *solution* is the entire problem-solving process, from the moment the problem is encountered, until we leave it as completed. The *answer* is something that appears along the way. While we insist on correct answers, it is the solution that is most important in the problem-solving process.

To help you teach problem solving, you might want to begin to create a Problem Deck. Take a package of large (5" × 9") file cards. Use a separate card for each problem. Write the problem on one side of the card. On the other side, write the solution or solutions, the strategy or strategies used to solve the problem, the correct answer, and where the problem may fit into your curriculum. Problems may be used in several ways:

1. As a means of introducing a topic
2. As a means of reviewing a topic taught earlier
3. As a means of summarizing a lesson just completed
4. As an enrichment of a topic taught

5. To dramatize a problem-solving technique
6. To demonstrate the power and beauty of mathematics

As you teach problem solving, you will encounter many problems that fit into one or more of these categories. Continue to add them to your problem cards collection. In this way, you will be constantly increasing your resource of problem-solving materials with which to teach mathematics.

We suggest that you read the book through, become familiar with all the strategies, practice them, and then begin to present them to your students. In this way, you and they can develop facility with the basic tools of problem solving.

In addition, we suggest that you begin to format more and more of your teaching in a problem-solving mode. That is, encourage your students to be creative in their approach to problems, encourage them to solve problems in a variety of ways, and encourage them to look for more than one method of solution to a problem. Have your students work together in small groups solving problems and communicating their ideas and work to others. The more students talk about problems and problem solving, the better they will become in this vital skill. Referring to the various problem-solving methods or strategies by name will ensure better and more efficient recollection when they are needed. Remember that the concept of metacognition (i.e., being aware of one's own thought processes) is an important factor in problem solving. Encouraging students to talk to themselves when tackling a problem is another way to help the students become aware of their problem-solving success.

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